Inequalities

Properties

Addition: (Similar rules hold for subtraction.)

- If $a < b$ then and only if $a + c < b + c$
- If $a \leq b$ then and only if $a + c \leq b + c$
- If $a > b$ then and only if $a + c > b + c$
- If $a \geq b$ then and only if $a + c \geq b + c$

Multiplication: (Similar rules hold for division.)

- If $c > 0$ and $a < b$ then $ac < bc$
- If $c > 0$ and $a > b$ then $ac > bc$
- If $c < 0$ and $a < b$ then $ac > bc$
- If $c < 0$ and $a > b$ then $ac < bc$

Interval Notation and Number Line Graphs

(a, b) = the set of all numbers between a and b

\[ \{x \mid a < x < b \} \]

- e.g., (-1, 3) = the set of all numbers between -1 and 4

\[ \{x \mid -1 < x < 3 \} \]

(a, b] = the set of all numbers greater than a and less than or equal to b

\[ \{x \mid a < x \leq b \} \]

- e.g., (-1, 3] = the set of all numbers greater than -1 and less than or equal to 4

\[ \{x \mid -1 < x \leq 3 \} \]

[a, b) = the set of all numbers greater than or equal to a and less than b

\[ \{x \mid a \leq x < b \} \]

- e.g., [-1, 3) = the set of all numbers between greater than or equal to -1 and less than 4

\[ \{x \mid -1 \leq x < 3 \} \]

(a, +\infty) = the set of all numbers greater than a

\[ \{x \mid x > a \} \]

- e.g., (-1, +\infty) = the set of all numbers greater than -1

\[ \{x \mid x > -1 \} \]
\([a, +\infty) = \text{the set of all numbers greater than or equal to } a\]
\[= \{x \mid x \geq a\}\]
\[\text{e.g., } [-1, +\infty) = \text{the set of all numbers greater than or equal to } -1\]
\[= \{x \mid x \geq -1\}\]

\((-\infty, b) = \text{the set of all numbers less than } b\]
\[= \{x \mid x < b\}\]
\[\text{e.g., } (-\infty, 3) = \text{the set of all numbers less than } 3\]
\[= \{x \mid x < 3\}\]

\((-\infty, b] = \text{the set of all numbers less than or equal to } b\]
\[= \{x \mid x \leq b\}\]
\[\text{e.g., } (-\infty, 3] = \text{the set of all numbers less than or equal to } 3\]
\[= \{x \mid x \leq 3\}\]

\((-\infty, +\infty) = \text{the set of all real numbers}\]
\[\ldots -2 \quad -1 \quad 0 \quad 1 \quad 2 \ldots \]

Absolute Value Inequalities \((a > 0)\)

\(|x| < a \text{ means that } x \text{ is a number whose distance from } 0 \text{ is less than } a\]
\[\text{or } -a < x < a\]
\[\text{e.g., } |x| < 2 \text{ means } -2 < x < 2\]

\(|x| \leq a \text{ means that } x \text{ is a number whose distance from } 0 \text{ is less than or equal to } a\]
\[\text{or } -a \leq x \leq a\]
\[\text{e.g., } |x| \leq 2 \text{ means } -2 \leq x \leq 2\]

\(|x| > a \text{ means that } x \text{ is a number whose distance from } 0 \text{ is greater than } a\]
\[\text{or } -a > x > a\]
\[\text{e.g., } |x| > 2 \text{ means } -2 > x > 2\]

\(|x| \geq a \text{ means that } x \text{ is a number whose distance from } 0 \text{ is greater than or equal to } a\]
\[\text{or } -a \geq x \geq a\]
\[\text{e.g., } |x| \geq 2 \text{ means } -2 \geq x \geq 2\]
To solve \( cx + d < a \) for \( x \)

1. Rewrite the inequality as \(-a < cx + d < a\)
2. Use the properties of inequalities to isolate \( x \) in the middle

\[
\begin{align*}
\text{e.g.} & & 2x - 5 < 7 & \quad \text{Rewrite the inequality.} \\
& & -7 < 2x - 5 < 7 & \quad \text{Add 5 to all three parts.} \\
& & -2 < 2x < 12 & \quad \text{Divide all three parts by 2 which is > 0} \\
& & -1 < x < 6 & \\
\text{Solution:} & & (-1, 6) &
\end{align*}
\]

The above steps will work for \( |cx + d| \leq a \) as well.

To solve \( cx + d > a \) for \( x \)

1. Rewrite the inequality as \( cx + d < -a \) or \( cx + d > a \)
2. Solve each inequality in step (1).
3. The solution is the union of the two solution sets.

\[
\begin{align*}
\text{e.g.} & & 2x - 5 > 7 & \quad \text{Rewrite the inequality.} \\
& & 2x - 5 < -7 \quad \text{or} \quad 2x - 5 > 7 & \quad \text{Add 5 to all three parts.} \\
& & 2x < -2 \quad \text{or} \quad 2x > 12 & \quad \text{Divide all three parts by 2 which is > 0} \\
& & x < -1 \quad \text{or} \quad x > 6 & \\
\text{Solution} & & (-\infty, -1) \cup (6, +\infty) &
\end{align*}
\]

The above steps will work for \( |cx + d| \geq a \) as well.

In a situation where the variable has a negative multiplier \( -cx \leq a \)

1. Rewrite the inequality as \(-a < -cx < a\)
2. When dividing the negative multiplier, the direction of all of the signs must change as well

\[
\begin{align*}
\text{e.g.} & & |-2x| < 6 & \quad \text{Rewrite the inequality} \\
& & -6 < -2x < 6 & \quad \text{Divide both sides by -2 and switch the direction of the signs} \\
& & 3 > x > -3 & \quad \text{Flip the entire equation around so that it will coincide with the number line} \\
& & -3 < x < 3 & \\
\text{Solution} & & (-3, 3) &
\end{align*}
\]

For all situations when a negative multiplier is involved, the direction of all of the signs must change when the multiplier is divided.