Methods of Solution of Selected Differential Equations
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Equations of Order One: $Mdx + Ndy = 0$

1. Separate variables.

2. M, N homogeneous of same degree:
   Substitute $y = vx$ or $x = vy$
   
   $dy = vdx + xdv$  
   $dx = vdy + ydv$

   and then separate variables.

3. Exact: \[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \]
   Solve \( \frac{\partial F}{\partial x} = M \) for F(x,y) including f(y) as constant term.
   Then compute \( \frac{\partial F}{\partial y} = N \) to find f(y).

   Solution is $F(x,y) = c$.

   Alternatively, start with \( \frac{\partial F}{\partial y} = N \).

4. Linear: $\frac{dy}{dx} + P(x)y = Q(x)$

   IF = \exp( \int Pdx)

   Multiply both sides of the equation by IF and result is exact.

   Left hand side will be \( \frac{d}{dx} (IF \cdot y) \)

5. The orthogonal trajectories to the family that has differential equation $Mdx + Ndy = 0$ have differential equation $Ndx - Mdy = 0$.

6. IF by inspection:

   Look for \( d(xy) = xdy + ydx \)
   \( d(\frac{y}{x}) = \frac{ydx - xdy}{x^2} \)
   \( d(\tan^{-1}\frac{y}{x}) = \frac{xdy - ydx}{x^2 + y^2} \)

   It may help to group terms of like degree.

7. IF for certain equations that are not homogeneous, not exact, and not linear:
   a. \( \frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = f(x) \), a function of x alone.

      IF = \exp( \int f(x) dx). Resulting equation is exact.

   b. \( \frac{1}{M} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = g(y) \), a function of y alone.

      IF = \exp( - \int g(y) dy). Resulting equation is exact.

8. Substitution suggested by the equation:
If an expression appears more than once, substituting a single variable for it may reduce the equation to a recognizable form.

9. Bernoulli: \( y' + P(x)y = Q(x)y^n \)
Substitute \( z = y^{1-n} \) and the resulting equation will be linear in \( z \).

10. Coefficients both linear:
\[
(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0
\]
Consider lines \( a_1x + b_1y + c_1 = 0 \)
\[
a_2x + b_2y + c_2 = 0
\]
a. If lines intersect at \( (h,k) \), substitute \( x = u + h, y = v + k \)
to get \( (a_1u + b_1v)du + (a_2u + b_2v)dv = 0 \) which is homogeneous.

b. If lines are parallel or coincide, use a substitution for recurring expression. (See 8)

**Linear Differential Equation:**
\[
b_0(x)\frac{d^n}{dx^n}y + b_1(x)\frac{d^{n-1}}{dx^{n-1}}y + \ldots + b_{n-1}(x)dy + b_n(x)y = R(x)
\]
1. The functions \( f_1, f_2, \ldots, f_n \) are linearly independent when
   \[
c_1f_1(x) + c_2f_2(x) + \ldots + c_nf_n(x) = 0 \text{ implies } c_1 = c_2 = \ldots = c_n = 0.
\]
2. The functions \( f_1, f_2, \ldots, f_n \) are linearly dependent if there exist constants \( c_1, c_2, \ldots, c_n \), not all zero, such that
   \[
c_1f_1(x) + c_2f_2(x) + \ldots + c_nf_n(x) = 0 \text{ identically on } a \leq x \leq b.
\]
3. The Wronskian of \( f_1, f_2, \ldots, f_n \) is
   \[
   \begin{vmatrix}
   f_1 & f_2 & f_3 & \ldots & f_n \\
   f_1' & f_2' & f_3' & \ldots & f_n' \\
   f_1'' & f_2'' & f_3'' & \ldots & f_n'' \\
   \vdots & \vdots & \vdots & \ddots & \vdots \\
   f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \ldots & f_n^{(n-1)}
   \end{vmatrix}
   
4. Theorem: If on \( (a,b) \), \( b_0(x) \neq 0, b_1, b_2, \ldots, b_n \) continuous,
   and \( y_1, y_2, \ldots, y_n \) are solutions of
   \[
b_0y^{(n)} + b_1y^{(n-1)} + \ldots + b_{n-1}y' + b_ny = 0
   \]
   then \( y_1, y_2, \ldots, y_n \) are linearly independent if and only if the
   Wronskian of \( y_1, y_2, \ldots, y_n \) is not zero on \( (a,b) \).

5. If \( y_1, y_2, \ldots, y_n \) are linearly independent solutions of the homogeneous equation,
   \[
b_0y^{(n)} + b_1y^{(n-1)} + \ldots + b_{n-1}y' + b_ny = 0
   \]
   then the general solution of this equation is
   \[
y = c_1y_1 + c_2y_2 + \ldots + c_ny_n.
   \]
6. The general solution of the equation
\[ b_0y^{(n)} + b_1y^{(n-1)} + \ldots + b_n y' + b_n y = R(x) \]
is \[ y = y_c + y_p, \] where \( y_c = c_1y_1 + c_2y_2 + \ldots + c_n y_n, \) the complementary function; \( y_1, y_2, \ldots, y_n \) are linearly independent solutions of the homogeneous equation; and \( c_1, c_2, \ldots, c_n \) are arbitrary constants; and \( y_p \) is any particular solution of the given nonhomogeneous equation.

7. A differential operator of order \( n \)
\[ A = a_0D^n + a_1D^{n-1} + \ldots + a_{n-1}D + a_n \] where \( D^k y = \frac{d^k y}{dx^k} \)

8. Properties of differential operators:
   a. If \( f(D) \) is a polynomial in \( D \), then \( f(D) \) \([e^{mx}] = e^{mx}f(m)\).
   b. If \( f(D) \) is a polynomial in \( D \) with constant coefficients, \( e^{ax}f(D)y = f(D-a) \) \([e^{ax}y]\) (“exponential shift”)
   c. \((D - m)^n(x^k e^{mx}) = 0 \) for \( k = 0, 1, \ldots, (n-1)\).

**Linear Equations with Constant Coefficients:**
\[ a_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_{n-1}y' + a_n y = R(x) \]
i.e., \( f(D)y = R(x) \)

1. The auxiliary equation associated with \( f(D)y = 0 \) is \( f(m) = 0 \).
   a. \( f(m) = 0 \) has distinct real roots \( m_1, m_2, \ldots, m_n \):
      \[ y_c = c_1e^{m_1 x} + c_2e^{m_2 x} + \ldots + c_n e^{m_n x} \]
   b. \( f(m) = 0 \) has repeated real roots. For each set of repetitions,
      \[ a, b, \ldots, b, \] the solutions are
      \[ c_1 e^{ax}, c_2 xe^{ax}, c_3 x^2 e^{ax}, \ldots, c_k x^{k-1} e^{ax} \]
   c. \( f(m) = 0 \) has distinct imaginary roots:
      For \( m = a \pm bi \), \( y = c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx \)
   d. \( f(m) = 0 \) has repeated imaginary roots. For example for
      \( a \pm bi, a \pm bi \), \( y = (c_1 + c_2x)e^{ax} \cos bx + (c_3 + c_4x)e^{ax} \sin bx \).
2. Method of undetermined coefficients:
   a. \( m_1, m_2, \ldots, m_n \) solutions of the auxiliary equation, so
      \[ y_c = c_1 y_1 + \ldots + c_n y_n \]
   b. Assuming \( R(x) \) is itself a particular solution of some homogeneous differential
      equation with constant coefficients which has roots \( m_1', m_2', \ldots, m_k' \) for its
      auxiliary equation. Write \( y_p \) from \( m_1', m_2', \ldots, m_k' \) being careful about any
      repetitions of \( m' \)-values with \( m \)-values. Substitute this \( y_p \) in the original equation,
      \( f(D)y = R(x) \) and equate corresponding coefficients.
   c. General solution: \( y = y_c + y_p \).

3. Solutions by inspection:
   a. If \( R(x) = \text{constant} \) and \( a_n \neq 0 \) then \( y_p = \frac{R(x)}{a_n} \)
   b. If \( R(x) = \text{constant} \) and \( a_n = 0 \) with \( y^{(k)} \) the lowest-order derivative that actually
      appears, then \( y_p = R(x) \cdot x^k \frac{k!}{a_{n-k}} \)

4. If \( y_1 \) is a particular solution of \( f(D)y = R_1(x) \) and \( y_2 \) is a particular solution of
   \( f(D)y = R_2(x) \), then \( y_p = y_1 + y_2 \) is a particular solution of \( f(D)y = R_1(x) + R_2(x) \).

**Linear Equations with Variable or Constant Coefficients**

\((b_0D^n + b_1D^{n-1} + \ldots + b_{n-1}D + b_n)y = R(x), b_i \) is not necessarily constant.

1. Reduction of order (d’Alembert): \( y'' + py' + qy = R \)
   If \( y = y_1 \) is a solution of the corresponding homogeneous equation:
   \( y'' + py' + qy = 0 \).
   Let \( y = vy_1, v \) variable, and substitute into original equation and simplify.
   Set \( v' = w \) and the resulting equation is a linear equation of first order in \( w \). Find the
   IF and solve for \( w \). Then since \( v' = w \), find \( v \) by integration. This gives \( y = vy_1 \).
2. Variation of parameters (Lagrange)
   
a. Order two: \( y'' + py' + qy = R(x) \)
   
   If \( y_c = c_1y_1 + c_2y_2 \), set \( y_p = A(x)y_1 + B(x)y_2 \), then find \( A \) and \( B \) so that
   
   \[
   \begin{cases}
   A'y_1 + B'y_2 = 0 \\
   A'y_1' + B'y_2' = R(x)
   \end{cases}
   
   Solve the system for \( A' \) and \( B' \), then for \( A \) and \( B \) by integration.
   
   Then \( y_p = A(x)y_1 + B(x)y_2 \).

b. Order three: \( y''' + py'' + qy' + r = s(x) \)
   
   If \( y_c = c_1y_1 + c_2y_2 + c_3y_3 \) then set \( y_p = A(x)y_1 + B(x)y_2 + C(x)y_3 \).
   
   \[
   \begin{cases}
   A'y_1 + B'y_2 + C'y_3 = 0 \\
   A'y_1' + B'y_2' + C'y_3' = 0 \\
   A'y_1'' + B'y_2'' + C'y_3'' = s(x)
   \end{cases}
   
   Solve the system for \( A', B', \) and \( C' \), then for \( A, B, \) and \( C \) by integration.
   
   Then \( y_p = A(x)y_1 + B(x)y_2 + C(x)y_3 \).

Inverse Differential Operators

1. Exponential shift: \( e^{ax}f(D)y = f(D-a)[e^{ax}y] \)

2. Evaluation of \( \frac{1}{f(D)}e^{ax} \)
   
   a. If \( f(a) \neq 0 \) then \( \frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)} \)
   
   b. If \( f(a) = 0 \) then \( \frac{1}{f(D)}e^{ax} = \frac{x^n e^{ax}}{n! \phi(a)} \), \( \phi(a) \neq 0 \).

3. Evaluation of \( (D^2 + a^2)^{-1}\sin ax \) and \( (D^2 + a^2)^{-1}\cos ax \)
   
   a. If \( a \neq b \), \( \frac{1}{D^2 + a^2} \sin bx = \frac{\sin bx}{a^2 - b^2} \)
   
   \( \frac{1}{D^2 + a^2} \cos bx = \frac{\cos bx}{a^2 - b^2} \)

   b. If \( a = b \), \( \frac{1}{D^2 + a^2} \sin ax = \frac{-x \cos ax}{2a} \)
   
   \( \frac{1}{D^2 + a^2} \cos ax = \frac{x \sin ax}{2a} \)
Laplace Transform

1. **Definition:** Laplace transform of \( F(t) = L\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt = f(s) \)

2. \( L \) is a linear transformation: \( c_1, c_2 \) constants
   \[ L\{c_1F_1 + c_2F_2\} = c_1L\{F_1\} + c_2L\{F_2\}. \]

3. Transforms of elementary functions.
   a. \( L\{e^{kt}\} = \frac{1}{s-k}, \quad s > k \)
   b. \( L\{\sin kt\} = \frac{k}{s^2 + k^2}, \quad s > 0 \)
   c. \( L\{\cos kt\} = \frac{s}{s^2 + k^2}, \quad s > 0 \)
   d. \( L\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0, \ n \) positive integer.

4. **Definition:** A function \( F(t) \) is **sectionally continuous** over \([a,b]\) if \([a,b]\) can be divided into a finite number of sub-intervals \([c,d]\) such that in each subinterval:
   (1) \( F(t) \) is continuous on \([c,d]\), and
   (2) \( \lim_{t \to c^+} F(t) \) and \( \lim_{t \to d^-} F(t) \) exist.

5. **Definition:** The function \( F(t) \) is of **exponential order** as \( t \to \infty \) if there exist constants \( M, b, \) and a fixed \( t_0 \) such that \( |F(t)| < Me^{bt} \) for \( t \geq t_0. \)
   a. **Note:** a bounded function is of exponential order as \( t \to \infty \)
   b. **Note:** if there is a \( b \) such that \( \lim_{t \to \infty} [e^{-bt}F(t)] \) exists, then \( F(t) \) is of exponential order as \( t \to \infty \).

6. **Definition:** A function of **Class A** is any function that is
   (1) sectionally continuous over every finite interval in the range \( t \geq 0, \) and
   (2) of exponential order as \( t \to \infty \).

7. **Theorem:** If \( F(t) \) is a function of Class A, then \( L\{F(t)\} \) exists.
8. Solution of initial value problems.

**Theorem:** If \( F(t), F'(t), \ldots, F^{(n-1)}(t) \) are continuous for \( t \geq 0 \) and of exponential order as \( t \to \infty \) and if \( F^{(n)}(t) \) is of Class A, then

\[
L\{F^{(n)}(t)\} = s^nL\{F(t)\} - \sum_{k=0}^{n-1} s^{n-1-k}F^{(k)}(0).
\]

In particular

\( n = 1: \ L\{F'(t)\} = sL\{F(t)\} - F(0) \)

\( n = 2: \ L\{F''(t)\} = s^2L\{F(t)\} - sF(0) - F'(0) \)

\( n = 3: \ L\{F'''(t)\} = s^3L\{F(t)\} - s^2F(0) - sF'(0) - F''(0) \)

**Theorem:** If \( F(t) \) is of exponential order as \( t \to \infty \) and \( F(t) \) is continuous for \( t \geq 0 \) except for a finite jump at \( t = t_1 \), and if \( F'(t) \) is of Class A, then from \( L\{F(t)\} = f(s) \), it follows that \( L\{F'(t)\} = sf(s) - F(0) - \exp(-st_1)[F(t_1^+) - F(t_1^-)] \)


**Theorem:** If \( F(t) \) is of Class A, then for every positive integer \( n \),

\[
\frac{d^n}{ds^n}f(s) = L\{(-t)^nF(t)\} \quad \text{where} \quad f(s) = L\{F(t)\}.
\]

10. Transform of a periodic function.

**Theorem:** If \( F(t) \) is periodic with period \( \omega \) and \( F(t) \) has a Laplace transform then \( L\{F(t)\} = \frac{\int_0^\omega e^{-st}F(t)\,dt}{1 - e^{-\omega s}} \)

11. **Definition:** If \( L\{F(t)\} = f(s) \) then \( F(t) \) is an **inverse transform** of \( f(s) \) and \( F(t) = L^{-1}\{f(s)\} \).

12. \( L^{-1} \) is a linear transformation.

13. **Theorem:** \( L^{-1}\{f(s)\} = e^{as}L^{-1}\{f(s-a)\} \).

**Gamma Function**

1. **Definition:** \( \Gamma(x) = \int_0^\infty e^{-\beta}\beta^{x-1}d\beta, \ x > 0 \).

2. **Theorem:** For all \( x > 0 \), \( \Gamma(x+1) = x\Gamma(x) \).

3. **Theorem:** \( \Gamma(n+1) = n! \) if \( n \) is a positive integer.

4. **Theorem:** \( L\{t^n\} = \frac{\Gamma(x+1)}{s^{x+1}}, \ s > 0, \ x > -1 \).