Methods of Solution of Selected Differential Equations

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Equations of Order One: Mdx + Ndy = 0

1. Separate variables.

2. M, N homogeneous of same degree:
   Substitute $y = vx$, $dy = vdx + xdv$ or $x = vy$, $dx = vdy + ydv$
   and then separate variables.

3. Exact: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$
   Solve $\dfrac{\partial F}{\partial x} = M$ for F(x,y) including f(y) as constant term.
   Then compute $\dfrac{\partial F}{\partial y} = N$ to find f(y).
   Solution is $F(x,y) = c$.
   Alternatively, start with $\dfrac{\partial F}{\partial y} = N$.

4. Linear: $\dfrac{dy}{dx} + P(x)y = Q(x)$ [IF=Integrating Factor]
   $IF = \exp(\int P\,dx)$
   Multiply both sides of the equation by IF and result is exact.
   Left hand side will be $\dfrac{d}{dx}(IF \bullet y)$

5. The orthogonal trajectories to the family that has differential equation $Mdx + Ndy = 0$ have differential equation $Ndx - Mdy = 0$.

6. IF by inspection:
   Look for $d(xy) = xdy + ydx$ $\quad d\left(\dfrac{y}{x}\right) = \dfrac{xdy - ydx}{x^2}$

   $d\left(\dfrac{x}{y}\right) = \dfrac{ydx - xdy}{y^2}$ $\quad d\left(\tan^{-1}\dfrac{y}{x}\right) = \dfrac{xdy - ydx}{x^2 + y^2}$

   It may help to group terms of like degree.

7. IF for certain equations that are not homogeneous, not exact, and not linear:
   a. If $\dfrac{1}{N}\left[\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right] = f(x)$, a function of x alone.
      $IF = \exp(\int f(x)\,dx)$. Resulting equation is exact.
   b. If $\dfrac{1}{M}\left[\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right] = g(y)$, a function of y alone.
      $IF = \exp(-\int g(y)\,dy)$. Resulting equation is exact.

8. Substitution suggested by the equation:

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If an expression appears more than once, substituting a single variable for it may reduce the equation to a recognizable form.

9. Bernoulli: $y' + P(x)y = Q(x)y^n$
Substitute $z = y^{1-n}$ and the resulting equation will be linear in z.

10. Coefficients both linear:
$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$
Consider lines $\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$

a. If lines intersect at (h,k), substitute $x = u + h$, $y = v + k$ to get $(a_1u + b_1v)du + (a_2u + b_2v)dv = 0$ which is homogeneous.

b. If lines are parallel or coincide, use a substitution for recurring expression. (See 8)

Linear Differential Equation:

$$b_0(x)\frac{d^ny}{dx^n} + b_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + b_{n-1}(x)\frac{dy}{dx} + b_n(x)y = R(x)$$

1. The functions $f_1, f_2, \ldots, f_n$ are linearly independent when $c_1f_1(x) + c_2f_2(x) + \ldots + c_nf_n(x) = 0$ implies $c_1 = c_2 = \ldots = c_n = 0$.

2. The functions $f_1, f_2, \ldots, f_n$ are linearly dependent if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that $c_1f_1(x) + c_2f_2(x) + \ldots + c_nf_n(x) = 0$ identically on $a \le x \le b$.

3. The Wronskian of $f_1, f_2, \ldots, f_n$ is
$$\begin{vmatrix} f_1 & f_2 & f_3 & \ldots & f_n \\ f_1' & f_2' & f_3' & \ldots & f_n' \\ f_1'' & f_2'' & f_3'' & \ldots & f_n'' \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \ldots & f_n^{(n-1)} \end{vmatrix}$$

4. Theorem: If on (a,b), $b_0(x) \neq 0$, $b_1, b_2, \ldots, b_n$ continuous, and $y_1, y_2, \ldots, y_n$ are solutions of
$b_0y^{(n)} + b_1y^{(n-1)} + \ldots + b_{n-1}y' + b_ny = 0$
then $y_1, y_2, \ldots, y_n$ are linearly independent if and only if the Wronskian of $y_1, y_2, \ldots, y_n$ is not zero on (a,b).

5. If $y_1, y_2, \ldots, y_n$ are linearly independent solutions of the homogeneous equation,
$b_0y^{(n)} + b_1y^{(n-1)} + \ldots + b_{n-1}y' + b_ny = 0$
then the general solution of this equation is $y = c_1y_1 + c_2y_2 + \ldots + c_ny_n$.

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6. The general solution of the equation
$b_0y^{(n)} + b_1y^{(n-1)} + \ldots + b_{n-1}y' + b_ny = R(x)$
is $y = y_c + y_p$, where $y_c = c_1y_1 + c_2y_2 + \ldots + c_ny_n$,
the complementary function; $y_1, y_2, \ldots, y_n$ are linearly independent solutions of the homogeneous equation; and $c_1, c_2, \ldots, c_n$ are arbitrary constants; and y_p is any particular solution of the given nonhomogeneous equation.

7. A differential operator of order n
$A = a_0D^n + a_1D^{n-1} + \ldots + a_{n-1}D + a_n$ where $D^ky = \dfrac{d^ky}{dx^k}$

8. Properties of differential operators:
   a. If f(D) is a polynomial in D, then $f(D)\,[e^{mx}] = e^{mx}f(m)$.

   b. If f(D) is a polynomial in D with constant coefficients,
   $e^{ax}f(D)y = f(D-a)\,[e^{ax}y]$ ("exponential shift")

   c. $(D - m)^n(x^ke^{mx}) = 0$ for $k = 0, 1, \ldots, (n-1)$.

Linear Equations with Constant Coefficients:
$a_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_{n-1}y' + a_ny = R(x)$
i.e., $f(D)y = R(x)$

1. The auxiliary equation associated with $f(D)y = 0$ is $f(m) = 0$.
   a. $f(m) = 0$ has distinct real roots $m_1, m_2, \ldots, m_n$:
   $y_c = c_1e^{m_1x} + c_2e^{m_2x} + \ldots + c_ne^{m_nx}$

   b. $f(m) = 0$ has repeated real roots. For each set of repetitions,
   say, $\overbrace{b, b, \ldots, b}^{k}$, the solutions are
   $c_1e^{bx}, c_2xe^{bx}, c_3x^2e^{bx}, \ldots, c_kx^{k-1}e^{bx}$

   c. $f(m) = 0$ has distinct imaginary roots:
   For $m = a \pm bi$, $y = c_1e^{ax}\cos bx + c_2e^{ax}\sin bx$

   d. $f(m) = 0$ has repeated imaginary roots. For example for
   $a \pm bi$, $a \pm bi$, $y = (c_1 + c_2x)e^{ax}\cos bx + (c_3 + c_4x)e^{ax}\sin bx$.

2. Method of undetermined coefficients:

   a. m_1, m_2, . . . , m_n solutions of the auxiliary equation, so
   $y_c = c_1y_1 + \ldots + c_ny_n$

   b. Assuming $R(x)$ is itself a particular solution of some homogeneous differential equation with constant coefficients which has roots m_1', m_2', . . . , m_k' for its auxiliary equation. Write y_p from m_1', m_2', . . . , m_k' being careful about any repetitions of m'-values with m-values. Substitute this y_p in the original equation, $f(D)y = R(x)$ and equate corresponding coefficients.

   c. General solution: $y = y_c + y_p$.

3. Solutions by inspection:
   a. If $R(x)$ = constant and $a_n \neq 0$ then $y_p = \dfrac{R(x)}{a_n}$
   b. If $R(x)$ = constant and $a_n = 0$ with $y^{(k)}$ the lowest-order derivative that actually appears, then $y_p = \dfrac{R(x) \bullet x^k}{k!\; a_{n-k}}$
4. If y_1 is a particular solution of $f(D)y = R_1(x)$ and y_2 is a particular solution of $f(D)y = R_2(x)$, then $y_p = y_1 + y_2$ is a particular solution of $f(D)y = R_1(x) + R_2(x)$.

Linear Equations with Variable or Constant Coefficients

$(b_0D^n + b_1D^{n-1} + \ldots + b_{n-1}D + b_n)y = R(x)$, b_i is not necessarily constant.

1. Reduction of order (d'Alembert): $y'' + py' + qy = R$
   If $y = y_1$ is a solution of the corresponding homogeneous equation:
   $$y'' + py' + qy = 0.$$
   Let $y = vy_1$, v variable, and substitute into original equation and simplify.
   Set $v' = w$ and the resulting equation is a linear equation of <u>first</u> order in w. Find the IF and solve for w. Then since $v' = w$, find v by integration. This gives $y = vy_1$.

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2. Variation of parameters (Lagrange)

a. Order two: $y'' + py' + qy = R(x)$
If $y_c = c_1y_1 + c_2y_2$, set $y_p = A(x)y_1 + B(x)y_2$, then find A and B so that this is a particular solution of the nonhomogeneous equation.

$$\begin{cases} A'y_1 + B'y_2 = 0 \\ A'y_1' + B'y_2' = R(x) \end{cases}$$

Solve the system for A' and B', then for A and B by integration.
Then $y_p = A(x)y_1 + B(x)y_2$.

b. Order three: $y''' + py'' + qy' + r = s(x)$
If $y_c = c_1y_1 + c_2y_2 + c_3y_3$ then set $y_p = A(x)y_1 + B(x)y_2 + C(x)y_3$.

$$\begin{cases} A'y_1 + B'y_2 + C'y_3 = 0 \\ A'y_1' + B'y_2' + C'y_3' = 0 \\ A'y_1'' + B'y_2'' + C'y_3'' = s(x) \end{cases}$$

Solve the system for A', B', and C', then for A, B, and C by integration.
Then $y_p = A(x)y_1 + B(x)y_2 + C(x)y_3$.

Inverse Differential Operators

1. Exponential shift: $e^{ax}f(D)y = f(D - a)\,[e^{ax}y]$

2. Evaluation of $\frac{1}{f(D)}e^{ax}$

a. If $f(a) \neq 0$ then $\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}$

b. If $f(a) = 0$ then $\frac{1}{\phi(D)(D-a)^n}e^{ax} = \frac{x^n e^{ax}}{n!\,\phi(a)}$, $\phi(a) \neq 0$.

3. Evaluation of $(D^2 + a^2)^{-1}\sin ax$ and $(D^2 + a^2)^{-1}\cos ax$.

a. If $a \neq b$, $\frac{1}{D^2 + a^2}\sin bx = \frac{\sin bx}{a^2 - b^2}$

$$\frac{1}{D^2 + a^2}\cos bx = \frac{\cos bx}{a^2 - b^2}$$

b. If $a = b$, $\frac{1}{D^2 + a^2}\sin ax = \frac{-x}{2a}\cos ax$

$$\frac{1}{D^2 + a^2}\cos ax = \frac{x}{2a}\sin ax$$

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Laplace Transform

1. Definition: Laplace transform of $F(t) = L\{F(t)\} = \int_0^\infty e^{-st}F(t)dt = f(s)$

2. L is a linear transformation: c_1, c_2 constants
   $L\{c_1F_1 + c_2F_2\} = c_1L\{F_1\} + c_2L\{F_2\}$.

3. Transforms of elementary functions.
   a. $L\{e^{kt}\} = \dfrac{1}{s-k}$, $s > k$

   b. $L\{\sin kt\} = \dfrac{k}{s^2+k^2}$, $s > 0$

      $L\{\cos kt\} = \dfrac{s}{s^2+k^2}$, $s > 0$

   c. $L\{t^n\} = \dfrac{n!}{s^{n+1}}$, $s > 0$, n positive integer.

4. Definition: A function $F(t)$ is sectionally continuous over $[a,b]$ if $[a,b]$ can be divided into a finite number of sub-intervals $[c,d]$ such that in each subinterval:
   (1) $F(t)$ is continuous on $[c,d]$, and
   (2) $\lim_{t\to c^+} F(t)$ and $\lim_{t\to d^-} F(t)$ exist.

5. Definition: The function $F(t)$ is of exponential order as $t\to\infty$ if there exist constants M, b, and a fixed t-value t_0 such that $|F(t)| < Me^{bt}$ for $t \geq t_0$.
   a. Note: a bounded function is of exponential order as $t\to\infty$
   b. If there is a b such that $\lim_{t\to\infty} [e^{-bt}F(t)]$ exists, then $F(t)$ is of exponential order as $t\to\infty$.

6. Definition: A function of Class A is any function that is
   (1) sectionally continuous over every finite interval in the range $t \geq 0$, and
   (2) of exponential order as $t\to\infty$.

7. Theorem: If $F(t)$ is a function of Class A, then $L\{F(t)\}$ exists.

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8. Solution of initial value problems.

Theorem: If $F(t)$, $F'(t), \ldots, F^{(n-1)}(t)$ are continuous for $t \geq 0$ and of exponential order as $t \to \infty$ and if $F^{(n)}(t)$ is of Class A, then $L\{F^{(n)}(t)\} = s^n L\{F(t)\} - \sum_{n=0}^{n-1} s^{n-1-k} F^{(k)}(0)$.

In particular

$n = 1$: $L\{F'(t)\} = sL\{F(t)\} - F(0)$.

$n = 2$: $L\{F''(t)\} = s^2 L\{F(t)\} - sF(0) - F'(0)$.

$n = 3$: $L\{F'''(t)\} = s^3 L\{F(t)\} - s^2 F(0) - sF'(0) - F''(0)$.

Theorem: If $F(t)$ is of exponential order as $t \to \infty$ and $F(t)$ is continuous for $t \geq 0$ except for a finite jump at $t = t_1$, and if $F'(t)$ is of Class A, then from $L\{F(t)) = f(s\}$, it follows that $L\{F'(t)\} = sf(s) - F(0) - \exp(-st_1)[F(t_1^+) - F(t_1^-)]$

9. Derivatives of transforms.

Theorem: If $F(t)$ is of Class A, then for every positive integer n,

$$\frac{d^n}{ds^n} f(s) = L\{(-t)^n F(t)\} \text{ where } f(s) = L\{F(t)\}.$$

10. Transform of a periodic function.

Theorem: If $F(t)$ is periodic with period ω and $F(t)$ has a Laplace transform

then $$L\{F(t))\} = \frac{\int_0^\omega e^{-st} F(t)\, dt}{1 - e^{-s\omega}}$$

11. Definition: If $L\{F(t)\} = f(s)$ then $F(t)$ is an inverse transform of $f(s)$ and $F(t) = L^{-1}\{f(s)\}$.

12. L^{-1} is a linear transformation.

13. Theorem: $L^{-1}\{f(s)\} = e^{-at} L^{-1}\{f(s - a)\}$.

Gamma Function

1. Definition: $\Gamma(x) = \int_0^\infty e^{-\beta} \beta^{x-1} d\beta$, $x > 0$.

2. Theorem: For all $x > 0$, $\Gamma(x + 1) + x\Gamma(x)$.

3. Theorem: $\Gamma(n + 1) = n!$ if n is a positive integer.

4. Theorem: $L\{t^x\} = \dfrac{\Gamma(x + 1)}{s^{x+1}}$, $s > 0$, $x > -1$.

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