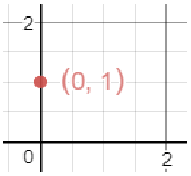
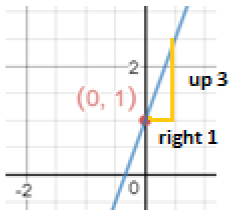
Quadratic Functions and Concavity

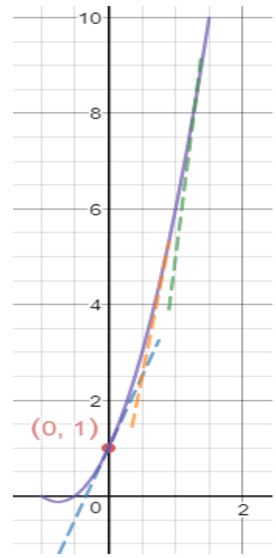
What does it mean for a graph to be concave up or concave down? Let's investigate.
Take, for example, the function $\boldsymbol{y = 2x^2 + 3x + 1}$. The "a" value is greater than zero.



We can start thinking about the graph at the vertical intercept of (0,1).



The graph, initially, behaves like the linear function $y = 3x + 1$; that is, the function initially increases at a rate of 3 output quantities for every change of 1 in the input quantity. This is what we mean when we explain the "b" value as the initial rate of change.



This constant rate of change does not persist, however. The output values are increasing at an increasing rate. Note that in the viewing window shown, the graph gets "steeper" and "steeper" as the input quantity increases from zero.
These changes are increasing at a rate of 2 times the "a" value. In this example the "a" value is 2, so 2*2=4. That initial rate of change is increasing at a rate of 4 per each increase of 1 in the input quantity. You can see this in the following table as the 2nd difference (Δ(Δy)).

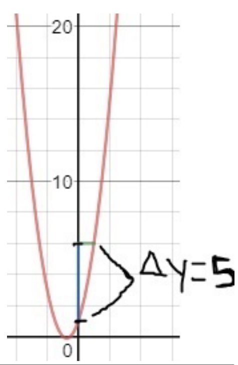
X	Y
-1	0
0	1
1	6
2	15
3	28

Δ y
1
5
9
13

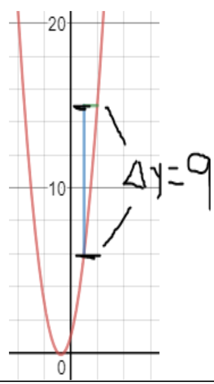
Δ (Δy)
4
4
4

Notice the ***a*** value in the equation $y = 2x^2 + 3x + 1$.

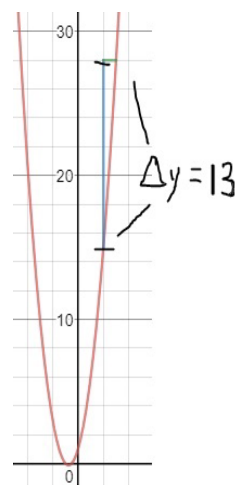
A is positive 2. The 2nd difference in the table is 4. Therefore, ***a*** is half of the change in the rate of change.



For an increase of 1 in the input, the output increased by 5.



For an increase of 1 in the input, the output increased by 9.



For an increase of 1 in the input, the output increased by 13.

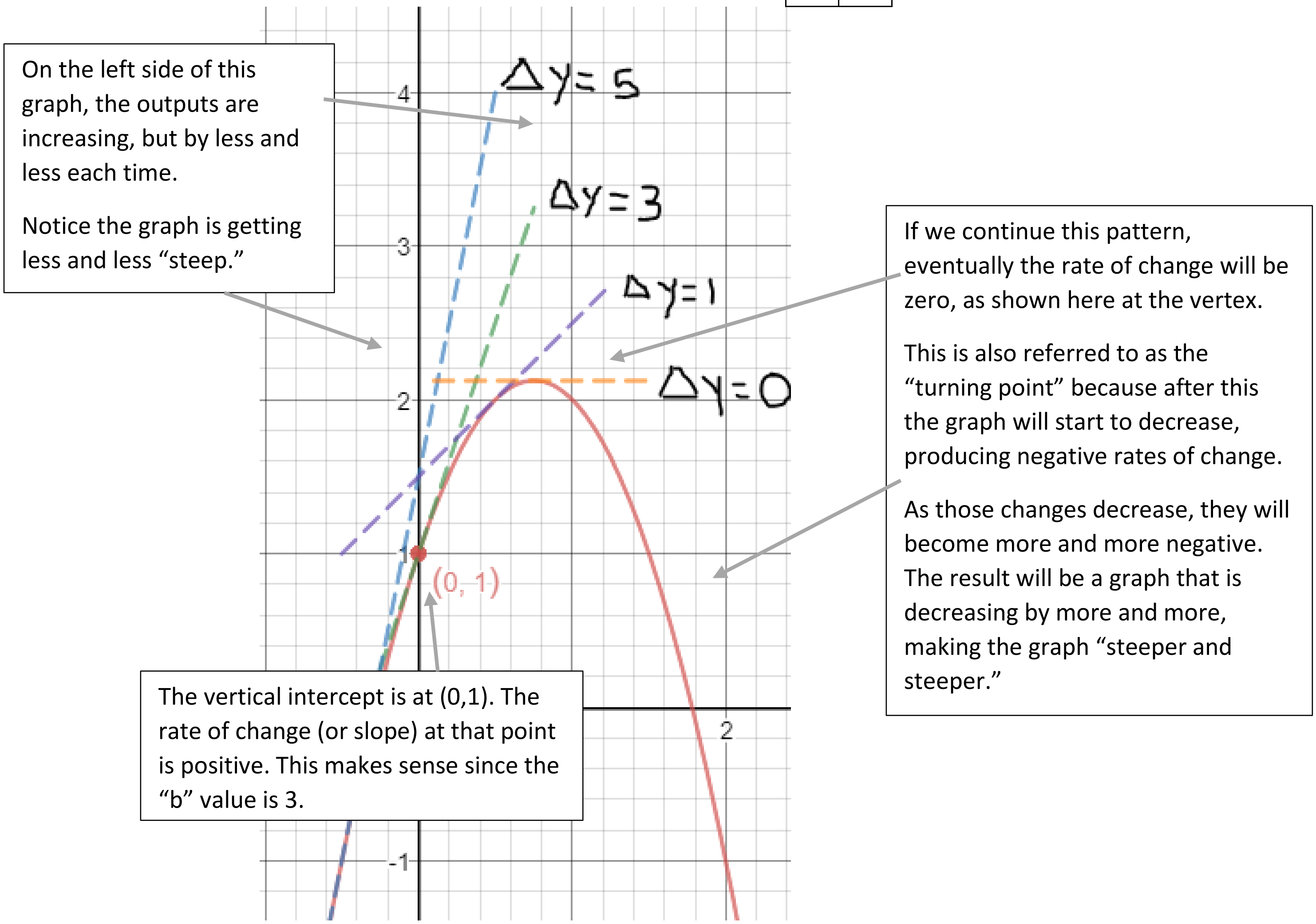
If the input quantity is decreased by 1, the rate of change is decreased by 2, and -2*2 = - 4. That is, we will see a decrease in the initial rate of change of 3 as the input quantity decreases. Imagine this continuing….eventually, the rate of change will be zero…then the rate of change will be negative. We see this negative rate of change to the left of the vertex. We see the rate of change of zero at the vertex, then the rate ever increasing after that! And that my friends, is how the parabola becomes concave up ☺.

Now let's explore a graph that is concave down. The function $y = -2x^2 + 3x + 1$ has an "a" value that is less than zero. On the left side of the graph, the outputs are increasing, but by less and less each time. Notice the vertical intercept is at (0,1). The rate of change (or slope) at that point is positive. That makes sense since the "b" value is 3. This initial rate of change does not stay constant, as previously discussed. The rate of change is getting smaller. We can say the function is changing at a decreasing rate. If we find the change in the rates of change, we will get a 2nd difference that is negative.

X	Y
-1	-4
0	1
1	2
2	-1
3	-8

Δ y
5
1
-3
-7

Δ (Δy)
- 4
- 4
- 4



Being concave up does not mean the graph increases, and being concave down does not mean the graph decreases. As shown with quadratic functions, there can be both increasing and decreasing portions for each concavity.

The concavity speaks more in depth about how the graph is changing. Concave up means the graph is changing at a rate that is going up. We usually say the function is increasing/decreasing "at an increasing rate." Concave down means the graph is changing at a rate that is going down, usually explained as the function is increasing/decreasing "at a decreasing rate." *The important thing to note about concavity is that individual rates of change are different and the emphasis is on how those rates of change are changing.*