

# MAT12X – Intermediate Algebra



## Workshop 3 – Rational Functions



CHANDLER-GILBERT  
COMMUNITY COLLEGE  
**LEARNING CENTER**

# Overview

## Workshop III

- Rational Functions
  - General Form
  - Domain and Vertical Asymptotes
  - Range and Horizontal Asymptotes
  - Inverse Variation
  - Practice Questions

General Form:  $h(x) = \frac{f(x)}{g(x)}$ , where  $g(x) \neq 0$ .

Rational functions are a ratio of one function to another. Each function has its own domain, but when any particular function is in the denominator of a rational function, that function cannot equal to zero. In the above example, this function is the  $g(x)$  function.

For example, if  $h(x) = \frac{3x}{x-2}$ , can we let  $x = 2$ ? Why not?

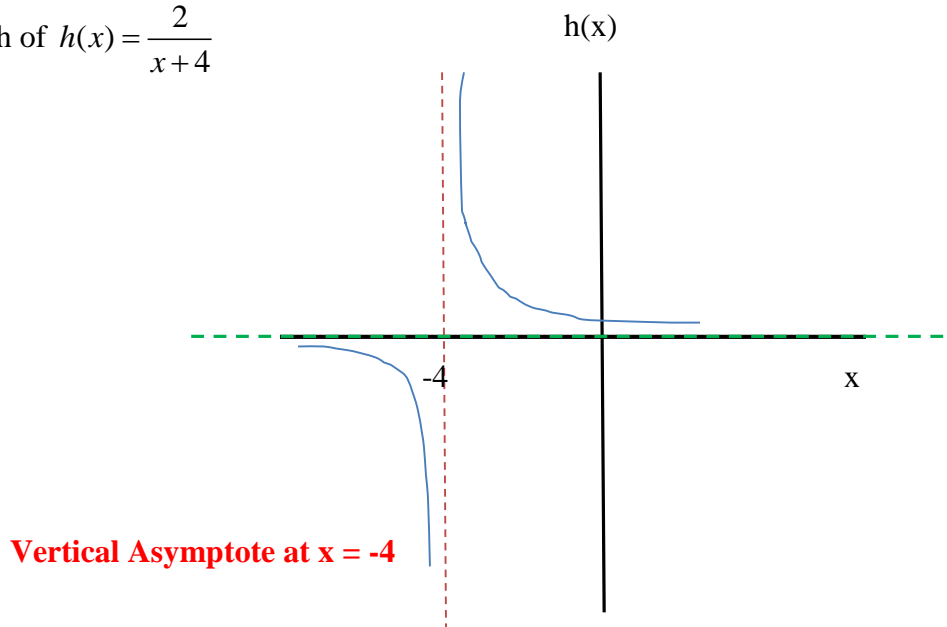
We cannot, because if we did the denominator would be zero which would result in the function being undefined. Since we cannot let  $x = 2$  in the  $h(x)$  function, we say the domain of  $h(x)$  is all real numbers except two. Or in interval notation:  $(-\infty, 2) \cup (2, \infty)$ . You try it! Find the domain for each of the following rational functions:

A.  $h(x) = \frac{2}{x+4}$  Domain:

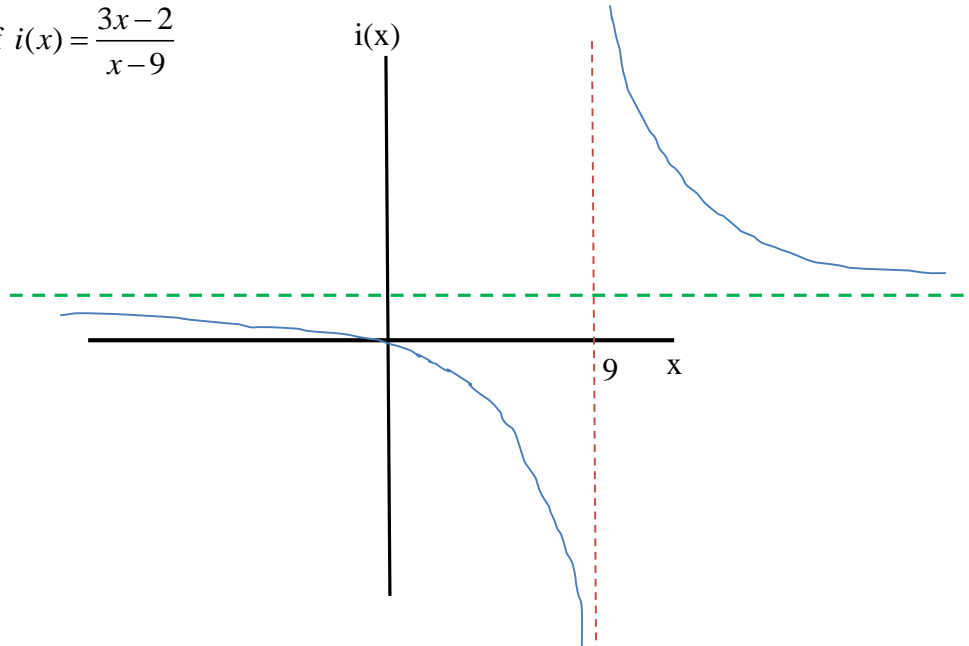
B.  $i(x) = \frac{3x-2}{x-9}$  Domain:

Once we know the domain of the rational function, any input values which will make the denominator zero we signify with a VERTICAL ASYMPTOTE! Because it is impossible to divide any number by zero, a vertical asymptote represents an input value that makes the denominator of a function zero. Looking at the above functions, the input values that are not in the domain become vertical asymptotes to let the reader of the graph know that we cannot choose these input values because if we do, we will then be dividing by zero which is undefined.

Graph of  $h(x) = \frac{2}{x+4}$



Graph of  $i(x) = \frac{3x-2}{x-9}$

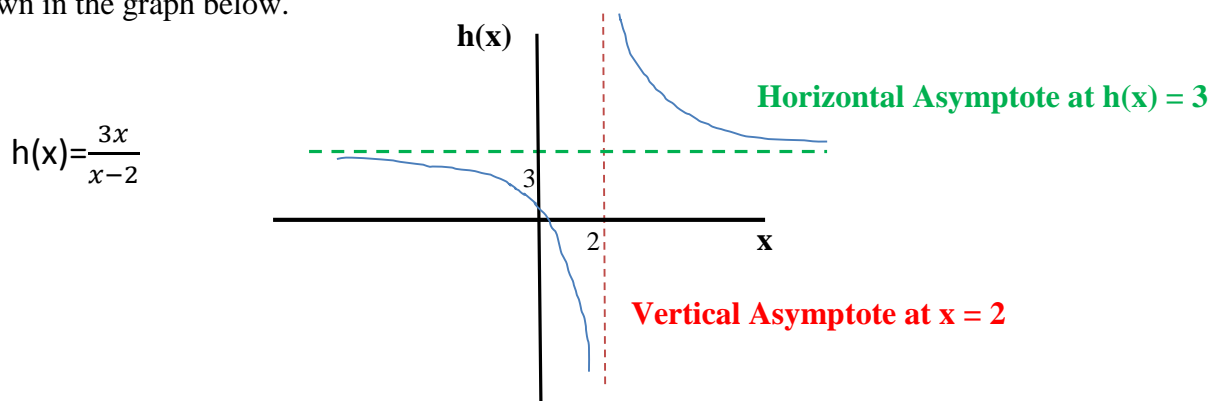


**Vertical Asymptote at  $x = 9$**

As we learn above, a real number “ $x$ ” is included in the domain if it gives us a real number as an output in a rational function  $h(x)$ . These output values represented by  $h(x)$  define the **range**. In other words, the **range** is the set of all real numbers of  $h(x)$  that correspond to existing real numbers of “ $x$ ” in the domain of the rational function  $h(x)$ .

Example: determine the range of the rational function  $h(x) = \frac{3x}{x-2}$

Solution: the range is all real numbers except three because if we graph this function in a graphing calculator, we can notice that the graph of  $h(x)$  gets values very close to  $h(x) = 3$ , but it never crosses it. This “ $h(x)=3$ ” is the Horizontal Asymptote of the rational function  $h(x)$  as it is shown in the graph below.



We can also represent the **range** of the function  $h(x)$  in interval notation  $(-\infty, 3) \cup (3, \infty)$ .

To practice this new concept, find the range of the following rational functions:

A.  $h(x) = \frac{2}{x+4}$  Range:

B.  $i(x) = \frac{3x-2}{x-9}$  Range:

As we showed in the graph of  $h(x)$  in the previous page, the horizontal asymptote is  $y=3$ . Therefore we can say that a **horizontal asymptote** of a rational function is an output value that the function approaches as the input goes to  $\pm\infty$ . It may or may not be included in the range. We determine the location of the **horizontal asymptote** by comparing the highest exponent “m” on the numerator with a leading coefficient “a” with the highest exponent “n” on the denominator with a leading coefficient “b” as follow:  $f(x) = \frac{ax^m}{bx^n}$

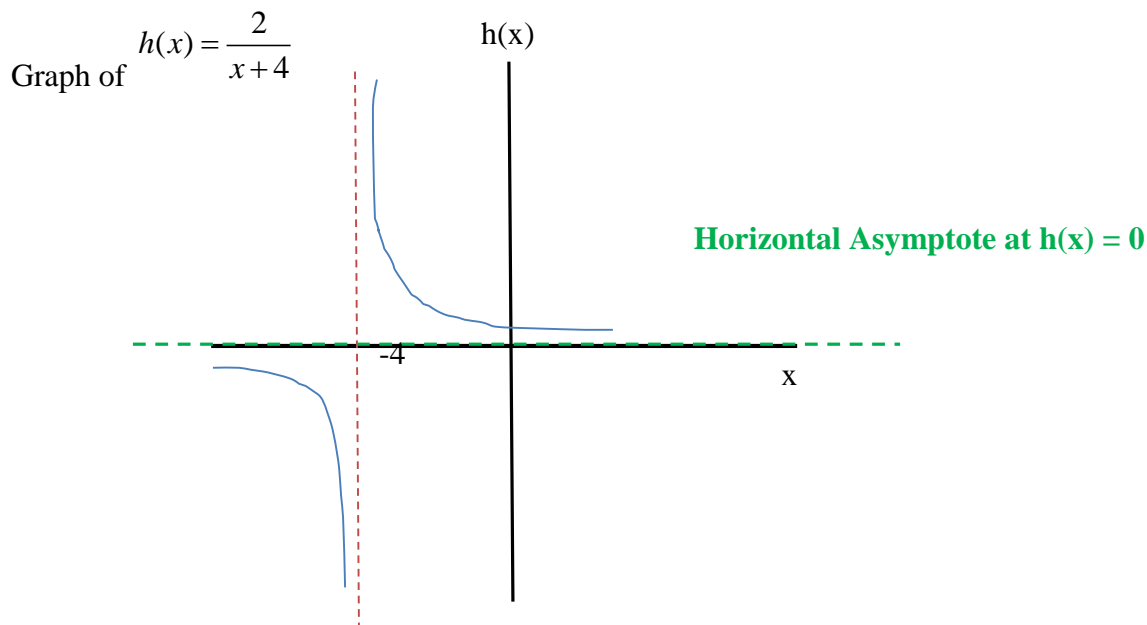
If  $m < n$ , there is a horizontal asymptote at  $f(x) = 0$

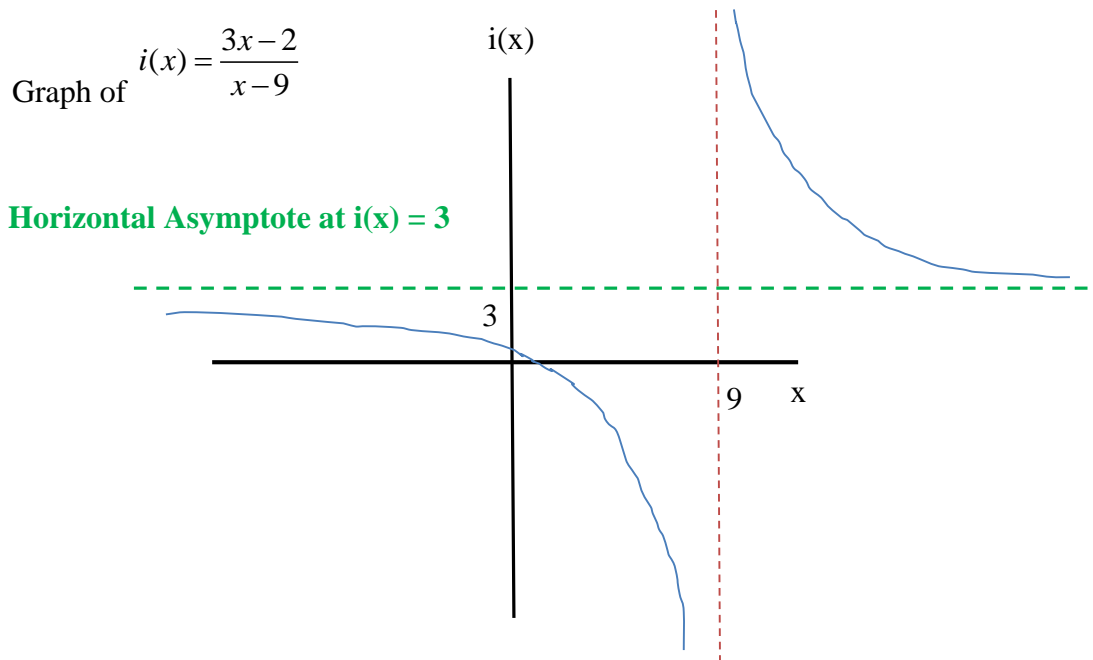
If  $m = n$ , there is a horizontal asymptote at  $f(x) = \frac{\text{leading coefficient on the numerator}}{\text{leading coefficient in the denominator}} = \frac{a}{b}$

For instance on the function  $h(x) = \frac{3x}{x-2}$

The highest exponent on the top is  $m = 1$ , and the highest exponent on the denominator is  $n = 1$ . Since  $m = n$ , the horizontal asymptote is at  $y = \frac{a}{b} = \frac{3}{1} = 3$  as it is shown in the graph of this function in the previous page.

Below we are can observe the graph of the previous functions A and B to identify the horizontal asymptote of each function if it exists:





Now that we learned the properties of rational functions, we are going to study another type of rational functions of the form  $y = \frac{k}{x^n}$ . This type of functions are called **Inverse Variation**

**Functions** where “k” is a nonzero real number (called constant of variation), the exponent “n” is a positive integer, and the variables “x” and “y” are inverse of each other which means that they change in an opposite manner because while one of them increases, the other variable decreases or vice versa.

Example:

“w” varies inversely as “p”, if  $w = 2$  and  $p = 100$ , find “w” when  $p = 50$ .

Solution:

**Step 1.** Write the form of the equation described in the problem

$$w = \frac{k}{p}$$

**Step 2:** Use the information given in the problem to find the value of k. In this case, you need to find k when  $w = 2$  and  $p = 100$ .

$$2 = \frac{k}{100}$$

$$100 * 2 = \frac{k}{100} * 100 \quad k = 200$$

**Step 3.** Rewrite the equation from step 1 using the value of “k” from step 2

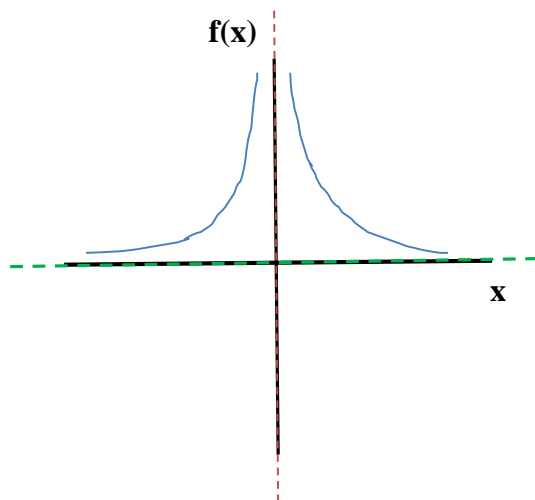
$$w = \frac{200}{p}$$

**Step 4.** Apply the formula from step 3 to find “w” when  $p = 50$

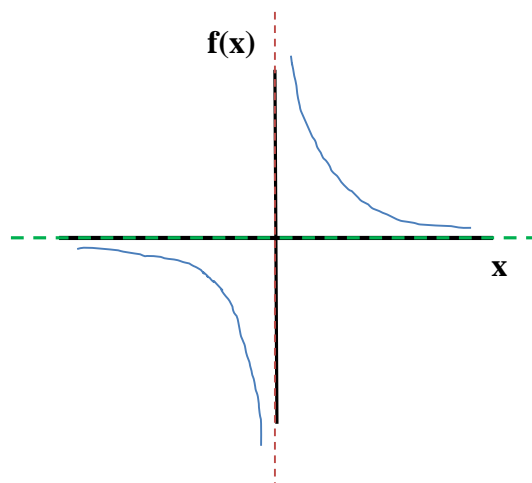
$$w = \frac{200}{50} = 4$$

Once we have learned inverse variation functions of the form  $f(x) = \frac{k}{x^n}$  and how we can use it to solve problems, we can learn the different shapes of the graphs for these types of functions as well as their properties shown below

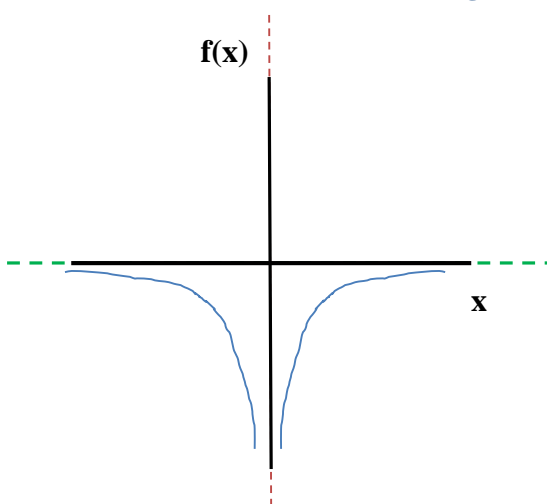
Case I.  $k > 0$ , and  $n$  is an even integer



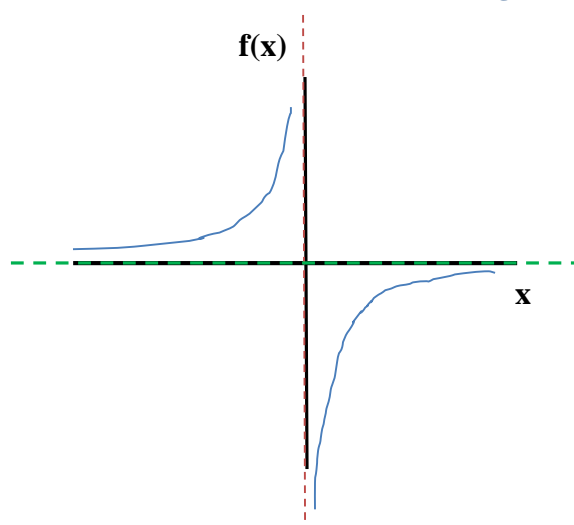
Case II.  $k > 0$ , and  $n$  is an odd integer



Case III.  $k < 0$ , and  $n$  is an even integer



Case IV.  $k < 0$ , and  $n$  is an odd integer



As we can observe in the four cases of these graphs, all of them have the following properties:

- The domain includes all real numbers except  $x = 0$ .
- A vertical asymptote at  $x = 0$ .
- A horizontal asymptote at  $f(x) = 0$ .
- There are no vertical or horizontal intercepts.
- There is no maximum or minimum function value.
- None of the functions are continuous at  $x = 0$ .

Answer the following questions for each of the problems 1-8:

- a. Find the domain
- b. Find the range
- c. Give the equation of the vertical asymptote
- d. Give the equation of the horizontal asymptote

1.  $f(x) = \frac{x-1}{x^2-4}$

2.  $g(x) = \frac{3x+6}{2x-1}$

3.  $f(x) = \frac{3}{x-5}$

4.  $g(x) = \frac{5}{x}$

5.  $h(x) = \frac{x-6}{x+3}$

6.  $f(x) = \frac{8}{x^2-4}$

7.  $g(x) = \frac{7}{x^3}$

Solve the following word problem

8. The time it takes you to get to campus varies inversely as your driving speed. Averaging 20 miles per hour in bad traffic, it takes you 1.5 hours to get to campus. How long would the trip take averaging 50 miles per hour?
9. The pressure of a perfect gas varies inversely as the volume. When the pressure of the gas is 15 atmospheres the volume is 5 liters. What will be the volume of the gas if the pressure is changed to 10 atmospheres?
10. The intensity of light produced by a light source varies inversely as the square of the distance from the source. If the intensity of light produced 3 feet from a light source is 750 foot-candles, find the intensity of light produced 5 feet from the same source.



Solutions:

*Problem 1.*

- a. Domain is all real numbers except -2 and 2.*
- b. Range is all real numbers.*
- c. There are two vertical asymptotes at  $x=-2$  and  $x=2$*
- d. There is a horizontal intercept at  $y=0$*

*Problem 2.*

- a. Domain is all real numbers except  $\frac{1}{2}$*
- b. Range is all real numbers except  $\frac{3}{2}$*
- c. There is a vertical asymptote at  $x=\frac{1}{2}$*
- d. There is an horizontal asymptote at  $y=\frac{3}{2}$*

*Problem 3.*

- a. Domain is all real numbers except 5*
- b. Range is all real numbers except 0*
- c. There is a vertical asymptote at  $x=5$*
- d. There is an horizontal asymptote at  $y=0$*

*Problem 4*

- a. Domain is all real numbers except 0*
- b. Range is all real numbers except 0*
- c. There is a vertical asymptote at  $x=0$*
- d. There is an horizontal asymptote at  $y=0$*

*Problem 5*

- a. Domain is all real numbers except -3*
- b. Range is all real numbers except 1*
- c. There is a vertical asymptote at  $x=-3$*
- d. There is an horizontal asymptote at  $y=1$*

*Problem 6*

- a. Domain is all real numbers except -2 and 2*
- b. Range is all real numbers except 0*
- c. There are two vertical asymptote at  $x=-2$  and  $x=2$*
- d. There is an horizontal asymptote at  $y=0$*

*Problem 7*

- a. Domain is all real numbers except 0
- b. Range is all real numbers except 0
- c. There is a vertical asymptote at  $x=0$
- d. There is an horizontal asymptote at  $y=0$

*Problem 8*

**Step 1:** Write the correct equation  $t = \frac{k}{s}$

**Step 2:** Use the information given in the problem to find the value of k. In this case, you need to find k when  $t = 1.5$  and  $s = 20$ .

$$1.5 = \frac{k}{20}$$

$$20 * 1.5 = \frac{k}{20} * 20$$

$$k = 30$$

**Step 3:** Rewrite the equation from step 1 substituting in the value of k found in step 2

$$t = \frac{30}{s}$$

**Step 4:** Use the equation found in step 3 and the remaining information given in the problem to answer the question asked. In this case, you need to find t when  $s = 50$ .

$$t = \frac{30}{50}$$

$$t = \frac{3}{5} = .6 \text{ hours}$$

**Problem 9**

**Step 1:** Write the correct equation  $p = \frac{k}{v}$

**Step 2:** Use the information given in the problem to find the value of k. In this case, you need to find k when  $p = 15$  and  $v = 5$

$$15 = \frac{k}{5}$$

$$5 * 15 = \frac{k}{5} * 5$$

$$k = 75$$

**Step 3:** Rewrite the equation from step 1 substituting in the value of k found in step 2

$$p = \frac{75}{v}$$

**Step 4:** Use the equation found in step 3 and the remaining information given in the problem to answer the question asked. In this case, you need to find v when p = 10.

$$\begin{aligned} 10 &= \frac{75}{v} \\ 10v &= 75 \\ v &= \frac{75}{10} = 7.5 \text{ liters} \end{aligned}$$

### Problem 10

**Step 1:** Write the correct equation  $i = \frac{k}{d^2}$

**Step 2:** Use the information given in the problem to find the value of k. In this case, you need to find k when i = 750 and d = 3.

$$\begin{aligned} 750 &= \frac{k}{3^2} \\ 750 * 3^2 &= k \\ k &= 6750 \end{aligned}$$

**Step 3:** Rewrite the equation from step 1 substituting in the value of k found in step 2

$$i = \frac{6750}{d^2}$$

**Step 4:** Use the equation found in step 3 and the remaining information given in the problem to answer the question asked. In this case, you need to find i when d = 5.

$$\begin{aligned} i &= \frac{6750}{5^2} \\ i &= 270 \text{ foot-candles} \end{aligned}$$